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To my parents,
Shoujin Wang and Guijia Sang,
for their 80th birthdays

Preface

The key point of Harnack's inequality is to compare values at two different points for positive solutions of a partial differential equation. This inequality was introduced by Harnack [21] in 1887 for harmonic functions on a Euclidean space, and was generalized by Serrin [46] in 1955 and Moser [34] in 1961 to solutions of elliptic or parabolic partial differential equations. Among many other applications, Harnack's inequality was used by Li and Yau [26] in 1986 to derive explicit heat kernel estimates, and by Hamilton [20] in 1993 to investigate the regularity of Ricci flows, which was then used in Perelman's proof of the Poincaré conjecture. All these Harnack inequalities are, however, dimension-dependent and thus invalid for equations on infinite-dimensional spaces.

In this book we aim to present a self-contained account of Harnack inequalities and applications for the semigroup of solutions to stochastic functional/partial differential equations. Since the associated Fokker–Planck equations are partial differential equations on infinite-dimensional spaces, the Harnack inequalities we are going to investigate are dimension-free. This is essentially different from the above-mentioned classical Harnack inequalities. Moreover, the main tool in our study is a new coupling method (i.e., coupling by change of measure) rather than the usual maximum principle in the literature of partial differential equations and geometric analysis.

The book consists of four chapters. In Chap. 1, we introduce a general theory concerning dimension-free Harnack inequalities, which includes the main idea of establishing Harnack inequalities and derivative formulas using coupling by change of measure, derivative formulas using the Malliavin calculus, links of Harnack inequalities to gradient estimates, and various applications of Harnack inequalities. In Chap. 2, we establish the Harnack inequality with power and the log-Harnack inequality for the semigroup associated to a class of nonlinear stochastic partial differential equations, which include stochastic generalized porous media/fast-diffusion equations as typical examples. The main tool is the coupling by change of measure introduced in Chap. 1. In Chap. 3, we investigate gradient estimates and Harnack inequalities for semilinear stochastic partial differential equations using coupling by change of measure, gradient estimates, and finite-dimensional approximations.

Chapter 4 is devoted to gradient estimates and Harnack inequalities for the segment solution of stochastic functional differential equations, using coupling by change of measure and the Malliavin calculus. To save space, applications of Harnack and shift Harnack inequalities presented in Chap. 1 are not restated in the other three chapters for specific models.

In this book we consider only stochastic functional/partial differential equations driven by Brownian motions. But the general theory introduced in Chap. 1 works also for stochastic differential equations driven by Lévy noises or the fractional Brownian motions; see [16, 17, 62, 63, 67, 76] and references therein. Materials of the book are mainly organized from the author's recent publications, including joint papers with colleagues who are gratefully acknowledged for fruitful collaborations. In particular, I would like to mention the joint work [3] with Marc Arnaudon and Anton Thalmaier, where the coupling by change of measure was used for the first time to establish the dimension-free Harnack inequality.

I would like to thank Xiliang Fan and Shaoqin Zhang for reading earlier drafts of the book and making corrections. I would also like to thank my colleagues from the probability groups of Beijing Normal University and Swansea University, in particular Mu-Fa Chen, Wenming Hong, Niels Jacob, Zenghu Li, Eugene Lytvynov, Yonghua Mao, Aubrey Truman, Jiang-Lun Wu, Chenggui Yuan, and Yuhui Zhang. Their kind help and constant encouragement provided me with an excellent working environment.

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Beijing, China Feng-Yu Wang

Contents

Chapter 1
A General Theory of Dimension-Free Harnack Inequalities

1.1 Coupling by Change of Measure and Applications

The dimension-free Harnack inequality was first established in [50] for the heat semigroup on Riemannian manifolds with curvature bounded below. To derive the same type inequality on manifolds with unbounded below curvature, the coupling by change of measure was introduced in [3]. Then it was applied to the study of Harnack-type inequalities and derivative formulas for solutions of various stochastic equations; see, e.g., [4, 5, 10, 15, 19, 27, 31, 43, 53, 54, 57, 58, 61, 64, 65, 66, 68, 74]. In this section we explain the main idea for the study of Harnack inequalities and derivative formulas in an abstract framework.

Definition 1.1. Let μ and ν be two probability measures on a measurable space $(E,\mathscr{B})$, and let X,Y be two E-valued random variables on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$.

(i) If the distribution of X is μ, while under another probability measure $\mathbb{Q}$ on $(\Omega,\mathscr{F})$ the distribution of Y is ν, we call (X,Y) a *coupling by change of measure* for μ and ν with changed probability $\mathbb{Q}$.

(ii) If μ and ν are distributions of two stochastic processes with path space E, a coupling by change of measure (X,Y) for μ and ν is called a coupling by change of measure for these two processes. In this case, X and Y are called the marginal processes of the coupling (X,Y).

Let $\mathscr{B}(E)$, $\mathscr{B}_b(E)$, and $\mathscr{B}_b^+(E)$ denote the sets of all measurable, bounded measurable, and nonnegative bounded measurable functions on E. When E is a topological space, we take $\mathscr{B}$ to be the Borel σ-field, and denote by $C(E), C_b(E)$, and $C_b^+(E)$ the set of all continuous, bounded continuous, and nonnegative bounded continuous functions on E.

For a family of probability measures $\{\mu_x : x\in E\}$, we define

$$Pf(x) = \int_E f(y)\mu_x(\mathrm{d}y) =: \mu_x(f), \quad f\in\mathscr{B}_b(E), x\in E. \tag{1.1}$$

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