Trends in Mathematics

Alexander Vasil'ev
Editor

Harmonic and Complex Analysis and its Applications

Birkhäuser

Trends in Mathematics

Alexander Vasil'ev
Editor

Harmonic and Complex Analysis and its Applications

Birkhäuser

Birkhäuser

Trends in Mathematics

Trends in Mathematics is a series devoted to the publication of volumes arising from conferences and lecture series focusing on a particular topic from any area of mathematics. Its aim is to make current developments available to the community as rapidly as possible without compromise to quality and to archive these for reference.

Proposals for volumes can be submitted using the Online Book Project Submission Form at our website www.birkhauser-science.com.

Material submitted for publication must be screened and prepared as follows:

All contributions should undergo a reviewing process similar to that carried out by journals and be checked for correct use of language which, as a rule, is English. Articles without proofs, or which do not contain any significantly new results, should be rejected. High quality survey papers, however, are welcome.

We expect the organizers to deliver manuscripts in a form that is essentially ready for direct reproduction. Any version of TEX is acceptable, but the entire collection of files must be in one particular dialect of TEX and unified according to simple instructions available from Birkhäuser.

Furthermore, in order to guarantee the timely appearance of the proceedings it is essential that the final version of the entire material be submitted no later than one year after the conference.

For further volumes:
http://www.springer.com/series/4961

Harmonic and Complex Analysis and its Applications

Alexander Vasil'ev
Editor

Birkhäuser

Editor
Alexander Vasil'ev
Department of Mathematics
University of Bergen
Bergen
Norway

ISBN 978-3-319-01805-8 ISBN 978-3-319-01806-5 (eBook)
DOI 10.1007/978-3-319-01806-5
Springer Cham Heidelberg New York Dordrecht London

Mathematics Subject Classification (2010): 13P15, 17B68, 17B80, 30C35, 30E05, 31A05, 31B05, 42C40, 46E15, 70H06, 76D27, 81R10

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.birkhauser-science.com)

Contents

Introduction

"Harmonic and Complex Analysis and its Applications" (HCAA) was a 5-year (2007–2012) European Science Foundation Programme whose aim was to explore and to strengthen the bridge between two scientific communities: analysts with broad backgrounds in complex and harmonic analysis and mathematical physics and specialists in physics and applied sciences. The programme was a multidisciplinary European activity uniting leading European scientists from these communities in 11 countries (Austria, Finland, Germany, Israel, Ireland, Luxembourg, Norway, Spain, Sweden, Switzerland, and UK) in a project of developing a coherent viewpoint on harmonic and complex analysis in the general context of mathematical physics. It coordinated actions for advancing harmonic and complex analysis and for increasing its application to challenging scientific problems. Particular topics considered by this programme included conformal and quasiconformal mappings, potential theory, Banach spaces of analytic functions and their applications to the problems of fluid mechanics, conformal field theory, Hamiltonian and Lagrangian mechanics, and signal processing. The programme had partnerships with other European and non-European networks. More on HCAA one can read at http://org.uib.no/hcaa/. The programme was steered by a committee

- *Alexander Vasil'ev* (Chairman) (University of Bergen, Norway);
- *Zoltan Balogh* (University of Bern, Switzerland);
- *Hans G. Feichtinger* (University of Vienna, Austria);
- *Stephen Gardiner* (University College Dublin, Ireland);
- *Björn Gustafsson* (Kungliga Tekniska Högskolan, Stockholm, Sweden);
- *Ilkka Holopainen and Olli Martio* (University of Helsinki, Finland);
- *John King and Linda Cummings* (University of Nottingham, UK);
- *Fernando Pérez-González* (Universidad de la Laguna, Tenerife, Spain);
- *Dierk Schleicher* (Jacobs University Bremen, Germany);
- *Martin Schlichenmaier* (Université du Luxembourg);
- *Lawrence Zalcman*, (Bar-Ilan University, Ramat-Gan, Israel).

Technical support was provided by an ESF (Physical and Engineering Sciences Unit) officer Mrs. *Catherine Werner* to whom we are all thankful for her kind timely help and assistance. This programme was very successful which is documented by several joint papers in highly ranked journals in line with many other ESF programmes in fundamental research and applications. We accept that it was a very useful and necessary instrument of funding in the European research area especially in the basic research, in particular in mathematics, in which there are not many other sources of external financing. At the beginning of the programme a kick-off conference was organized in Norway in 2007 and a volume of proceedings appeared in [1]. During the programme we started a new journal [2] under the same name that publishes current research results as well as selected high-quality survey articles in real, complex, harmonic, and geometric analysis originating and/or having applications in mathematical physics. The journal promotes dialog among specialists in these areas. After the time period of HCAA had finished, the Steering Committee decided to edit a special volume of surveys reflecting important research lines represented by the participants of the programme. The committee served as the Advisory Board of this volume.

We would like to acknowledge the efforts of all participants of the programme HCAA, in particular, the authors of this volume. We hope that it will be interesting and useful for professionals and novices in analysis and mathematical physics. We strongly believe that graduate students will be one of the target audiences of this book. Browsing the volume, the reader undoubtedly notices that the scope of the programme being rather broad exhibits many interrelations between the various contributions, which can be regarded as different facets of a common theme. We hope that this volume will enrich the further development in analysis and its prosperous interaction with mathematical physics and applied sciences.

Bergen, Norway Alexander Vasil'ev

References

1. B. Gustafsson, A. Vasil'ev (eds.), *Analysis and Mathematical Physics*, Trends in Mathematics (Birkhäuser Verlag, Boston, 2009)
2. B. Gustafsson, A. Vasil'ev (eds.), *Analysis and Mathematical Physics* (Birkhäuser Verlag, Boston). ISSN:1664–2368

Function Spaces of Polyanalytic Functions

Luis Daniel Abreu and Hans G. Feichtinger

Abstract This article is meant as both an introduction and a review of some of the recent developments on Fock and Bergman spaces of polyanalytic functions. The study of polyanalytic functions is a classic topic in complex analysis. However, thanks to the interdisciplinary transference of knowledge promoted within the activities of HCAA network it has benefited from a cross-fertilization with ideas from signal analysis, quantum physics, and random matrices. We provide a brief introduction to those ideas and describe some of the results of the mentioned cross-fertilization. The departure point of our investigations is a thought experiment related to a classical problem of multiplexing of signals, in order words, how to send several signals simultaneously using a single channel.

Keywords Gabor frames • Landau levels • Polyanalytic spaces

L.D. Abreu (✉)
Acoustic Research Institute, Austrian Academy of Sciences, Wohllebengasse 12–14, Wien A-1040, Austria

CMUC, Department of Mathematics, University of Coimbra, Portugal
e-mail: daniel@mat.uc.pt

H.G. Feichtinger
NuHAG, Faculty of Mathematics, University of Vienna, Nordbergstraße 15, A-1090 Wien, Austria
e-mail: hans.feichtinger@univie.ac.at

A. Vasil'ev (ed.), *Harmonic and Complex Analysis and its Applications*, Trends in Mathematics, DOI 10.1007/978-3-319-01806-5_1,

1 Introduction

1.1 Definition of a Polyanalytic Function

Among the most widely studied mathematical objects are the solutions of the Cauchy–Riemann equation

$$\partial_{\overline{z}} F(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right) F(x + i\xi) = 0,$$

known as *analytic functions*. The properties of analytic functions are so remarkable that, at a first encounter, they are often perceived as "magic." However, the analyticity restriction is so strong that it created a prejudice against non-analytic functions, which are often perceived as unstructured and bad behaved objects and therefore not worthy of further study. Nevertheless, there are non-analytic functions with significant structure and with properties reminiscent of those satisfied by analytic functions.

Such nice non-analytic functions are called *polyanalytic functions.*

A function $F(z, \overline{z})$, defined on a subset of $\mathbb{C}$, and satisfying the generalized Cauchy–Riemann equations

$$(\partial_{\overline{z}})^n F(z, \overline{z}) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)^n F(x + i\xi, x - i\xi) = 0, \tag{1}$$

is said to be *polyanalytic of order* $n - 1$. It is clear from (1) that the following polynomial of order $n - 1$ in $\overline{z}$

$$F(z, \overline{z}) = \sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z), \tag{2}$$

where the coefficients $\{\varphi_k(z)\}_{k=0}^{n-1}$ are analytic functions is a polyanalytic function of order $n - 1$. By solving $\partial_{\overline{z}} F(z, \overline{z}) = 0$, an iteration argument shows that every $F(z, \overline{z})$ satisfying (1) is indeed of the form (2). Some fundamental properties of analytic functions cease to be true for polyanalytic functions. For instance, a simple polyanalytic function of order 1 is

$$F(z, \overline{z}) = 1 - |z|^2 = 1 - z\overline{z}.$$

Since

$$\partial_{\overline{z}} F(z, \overline{z}) = -z \text{ and } (\partial_{\overline{z}})^2 F(z, \overline{z}) = 0,$$

the function $F(z, \overline{z})$ is not analytic in z, but is *polyanalytic of order one*. This simple example already highlights one of the reasons why the properties of polyanalytic functions can be different of those enjoyed by analytic functions: they can vanish on closed curves without vanishing identically, while analytic functions cannot even vanish on an accumulation set of the complex plane! Still, many properties of analytic functions have found an extension to polyanalytic functions, often in a nontrivial form, as we shall see later on in a few examples.

1.2 What Are Polyanalytic Functions Good for?

Imagine some application of analytic functions. By definition, they allow to represent the objects of our application as a function of z (because the function is analytic). We may want to represent the object to obtain a nice theory, we may want to store the information contained in the object and send it to someone. Whatever we want to do, we will always end up with a representation involving powers of z (because the functions are analytic). Not that bad, since we have an infinite number of them. However, several applications of mathematics, like quantum mechanics and signal analysis, require infinite dimensions for their theoretical formulation. And when we build a model in the complex plane using analytic functions, all the powers of z are taken.

What if we want to build several models simultaneously for the same plane? Introducing an extra complex variable will bring us the complications related to the study of analytic functions in several complex variables. If $\mathbb{C}$ is not enough for some models, $\mathbb{C}^2$ may seem too much to handle if we want to keep the mathematical problems within a tangible range. One is tempted to ask if there is something in between, but it may seem hard to believe that it is possible to "store" more information in a complex plane without introducing an extra independent variable.

Enter the world of polyanalytic functions!

We are now allowed to use powers of z and $\overline{z}$. This introduces an enormous flexibility. Consider the Hilbert space $\mathcal{L}_2(\mathbb{C})$ of all measurable functions equipped with the norm

$$\|F\|^2_{\mathcal{L}_2(\mathbb{C})} = \int_{\mathbb{C}} |F(z)|^2 \, e^{-\pi|z|^2} \, d\mu(z). \tag{3}$$

It is relatively easy to observe, using integration by parts (see formula (6) below) that, given an analytic function $F(z) \in \mathcal{L}_2(\mathbb{C})$, the function

$$F'(z) - \pi \overline{z} F(z)$$

is orthogonal to $F(z)$. We can create several subspaces of $\mathcal{L}_2(\mathbb{C})$ by multiplying elements of the Fock space of analytic functions by a power of $\overline{z}$. If we consider the sum of all such spaces, we obtain the whole $\mathcal{L}_2(\mathbb{C})$. We can do even better: by proper combination of the powers of z and $\overline{z}$ we can obtain an orthogonal decomposition of $\mathcal{L}_2(\mathbb{C})$! This fact, first observed by Vasilevski [76], is due to the following: the polynomials

$$e_{k,j}(z,\overline{z}) = e^{\pi|z|^2} (\partial_z)^k \left[e^{-\pi|z|^2} z^j \right]$$

are orthogonal in both the index j and k and they span the whole space $\mathcal{L}_2(\mathbb{C})$ of square integrable functions in the plane weighted by a gaussian $e^{-\pi|z|^2}$. For every k, we have thus a "copy" of the space of analytic functions which is orthogonal to any of the other copies. It is a remarkable fact that every polyanalytic function of order n can be expressed as a combination of the polynomials $\{e_{k,j}(z,\overline{z})\}_{k<n,j\in\mathbb{N}}$. Thus, we can work simultaneously in n planes keeping the number of degrees of freedom of each one intact. We will see in this paper how this fact can be put in good use, notably in the analysis of the higher Landau levels and in the multiplexing of signals (analysis of several signals simultaneously).

1.3 Some Historical Remarks

Polyanalytic functions were for the first time considered in [55] by the Russian mathematician G. V. Kolossov (1867–1935) in connection with his research on elasticity. This line of research has been developed by his student Muskhelishvili and the applications of polyanalytic functions to problems in elasticity are well documented in his book [63].

Polyanalytic function theory has been investigated intensively, notably by the Russian school led by Balk [15]. More recently the subject gained a renewed interest within operator theory and some interesting properties of the function spaces whose elements are polyanalytic functions have been derived [16,17]. A new characterization of polyanalytic functions has been obtained by Agranovsky [9]. The two visionary papers of Vasilevski [76] and [75] had a profound influence in what we will describe. In the beginning, our investigations in the topic were motivated by applications in signal analysis, in particular by the results of Gröchenig and Lyubarskii on Gabor frames with Hermite functions [44, 45], but soon it was clear that Hilbert spaces of polyanalytic functions lie at the heart of several interesting mathematical topics. Remarkably, they provide explicit representation formulas for the functions in the eigenspaces of the Euclidean Laplacian with a magnetic field, the so-called *Landau levels*. It is also worth of historical remark that, in 1951, Richard Feynman has obtained formulas very similar to those involving

Gabor transforms with Hermite functions in his work on quantum electrodynamics [37]. Precisely the same functions have been used by Daubechies and Klauder in their formulation of Feynman integrals [25]. The historically conscious reader may recognize in polyanalytic function theory some of the eclectic flavor emblematic of the mathematics oriented to signal analysis and quantum mechanics, something particularly notorious in the body of mathematics which became known as the Bell papers of the 1960s (see the review [73]) and in the advent of wavelets and coherent states [10, 24].

The following are the main books that we used as sources in important topics which are briefly outlined in this survey. Balk's book [15] is still an authoritative reference for the function theoretical aspects of polyanalytic functions. Standard references for Fock and Bergman spaces are [29, 51, 79] and for the applications of such spaces in sampling we have followed [72]. Our notations and essential facts about Gabor analysis are extracted from Gröchenig's book [42]. We used [10] as a reference for coherent states and Daubechies classical monograph [24] for wavelets and general frame theory. An introduction to Determinantal Point Processes can be found in [18].

1.4 Outline

Our main goal is to highlight the connections between different topics. We try to present the material in a such a way that the reader can gain from the time-frequency point of view, even if it is the case of a reader who is not familiar with the basic concepts of time-frequency analysis. Thus, the basic concepts are presented and a special attention is given to those particular regions of knowledge where two different mathematical topics intersect.

We have organized the paper as follows. We start with a section on the Hilbert space theory of polyanalytic Fock spaces. This includes the description of the theoretical multiplexing device which is the basic signal analytic model for our viewpoint. The third section explains how the topic connects to time-frequency analysis, more precisely, to the theory of Gabor frames with Hermite functions. Then we make a review of the basic facts of the L^p theory of polyanalytic Fock spaces for $p \neq 2$ and of the polyanalytic Bargmann transform in modulation spaces. We review some physical applications in Sect. 4, namely the interpretation of the so-called true polyanalytic Fock spaces as the eigenspaces of the Euclidean Landau Hamiltonian with a constant magnetic field. In Sect. 6 we provide a brief introduction to the polyanalytic Ginibre ensemble. The last section is devoted to hyperbolic analogues of the theory, where wavelets and the Maass Laplacian play a fundamental role.