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Marco Bramanti

An Invitation to Hypoelliptic Operators and Hörmander's Vector Fields

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Preface

Hörmander's operators are an important class of linear elliptic-parabolic degenerate partial differential operators with smooth coefficients, which have been intensively studied since the late 1960s and are still an active field of research. This text is based on the notes that I wrote for a short course held at the Department of Mathematics of *Politecnico di Milano* in February–March 2012. The audience consisted of a group of colleagues and Ph.D. students working on PDEs. In accordance with the style of that course, this text is written for people who are interested in beginning to do research in this area, or are just curious about it, and look for a *general overview* of this field of research, with its motivations and problems, some of its fundamental results, and at least some of its recent lines of development. In this text *proofs* are almost completely absent, which is an uncommon feature for a mathematical booklet, and therefore needs some justification, which will appear from the rest of this introduction.

Let me first sketch the plan of these notes. Chapters 1 and 2 deal, in two different ways, with *motivations* for studying Hörmander's operators. (Actually, the quest of motivation is a leitmotiv of these notes). Chapter 1 explains the PDE context at the origin of the concept of Hörmander's operators, while Chap. 2 focuses on the relevance in other areas of pure or applied mathematics of specific classes of partial differential operators which are actually of Hörmander's type. Chapters 3 and 4 discuss some fundamental ideas and results in this area, dating back to the 1970s and 1980s, which everyone interested in doing research, or studying contemporary research papers about Hörmander's operators, needs to know. More specifically, Chap. 3, perhaps the technical core of this text, deals with the theme of a-priori estimates, in the suitable Sobolev spaces, for Hörmander's operators. This involves the concept of homogeneous groups, the construction of fundamental solutions, the use of abstract singular integral theories, and the development of suitable algebraic and differential geometric tools. Chapter 4 deals with the geometry of the vector fields which appear in the definition of Hörmander's operators, the concept of distance induced by a system of vector fields, and related problems. Actually, the study of *geometry of Hörmander's vector fields* is nowadays an independent field of research with respect to the study of *second order PDEs of Hörmander's type*. These notes are mainly focussed on the second theme, with a touch on the first one mainly as a tool for the study of the second one. However, in Chap. 4 I have at least tried to give some of the motivations for the study of the geometry of Hörmander's vector fields, besides its applications to PDEs. Finally, Chap. 5 presents an overview of some of the developments of the theory of Hörmander's operators in the 1990s and 2000s. Here the choice of the topics particularly reflects my personal interests. As we will see, the evolution of this area has mainly consisted of the study of classes of operators which are no longer of Hörmander type, strictly speaking, but are structured on Hörmander's vector fields, in several senses, and also contain some nonsmooth ingredients, which poses new problems. Again, I have always tried to give some motivations for the study of the particular classes of operators which are considered.

Let me now try to justify the style of this text. Motivation for the study of Hörmander's operators and vector fields involve, among other areas, systems of stochastic differential equations, the theory of functions of several complex variables, geometric control theory and nonholonomic mechanics: a wide range of subjects, which is impossible to give account of in a few pages. The fundamental ideas and results which still now constitute the basic tools to work on Hörmander's operators appeared in the literature in a small number of very important papers, which are very technical, long, and often not easy to read. A detailed exposition of the contents of those papers, in the style of a graduate course, would require several hundreds of pages. I think that both for a person who is just curious about this area and for one who wants to begin doing active research in it, acquiring from the very beginning a general picture of the landscape can be of great help. Doing this in a limited time requires avoiding proofs. The study of proofs and techniques is a, clearly unavoidable, second step which a person will take, starting with the specific techniques involved in the specific problem he/she wants to attack.

Any comment on this book will be appreciated. Please, write to: marco.bramanti@polimi.it

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Contents

Chapter 1
Hörmander's Operators: What they are

1.1 The Context of Distribution Theory

In 1950 Laurent Schwartz was awarded with the Fields Medal for his creation of the *theory of distributions* (see [16, 17]). Since every distribution is infinitely differentiable but, on the other hand, only the product of a *smooth* function with a distribution is generally a distribution, this theory is a natural framework for the study of *linear partial differential equations with smooth coefficients*. For this class of equations the theory allows to give a very general definition of solution, which turns out to be appropriate under several regards. For instance, in spite of the apparent weakness of the concept of distributional solution, this notion is capable of distinguishing between a solution to $Lu = 0$ and a fundamental solution of the operator L, a distinction which the notion of "equality almost everywhere" fails to reveal. Distribution theory provided for the following decades the conceptual framework for the study of general properties of partial differential operators with smooth coefficients.[1] One of the leaders in this field of research was Lars Hörmander, who won the Fields Medal in 1962 for his deep study of linear PDEs with smooth coefficients. The book [7] represents an account of Hörmander's results in this field up to that time.

Let us consider a linear differential operator of order m, with (real or complex) coefficients, defined and infinitely differentiable on the whole $\mathbb{R}^n$ or in some domain. If we want to establish some general properties, holding independently of the kind of equation (elliptic, hyperbolic...), we cannot study a boundary or initial value problem. Instead, two basic questions are:

Does equation $Lu = f$ possess any solution?

If $Lu = f$ and f is smooth, is u smooth?

The first question introduces the notion of *solvability*, the second one that of *hypoellipticity*.

[1] An interesting account of the early impact of the theory of distributions on the mathematical environment is given by Lars Gårding in [6, Chap. 12].

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1.2 Local Solvability

Definition 1 *(See [5, p. 85]) An operator L is said locally solvable at x_0 if there exists a neighborhood $\mathcal{U}$ of x_0 such that for any $f \in C_0^\infty(\mathcal{U})$, equation $Lu = 0$ has a (distributional) solution $u \in \mathcal{D}'(\mathcal{U})$.*

Recall that the fundamental solution $\Gamma_y(\cdot)$ of an operator L with pole y is, by definition, a distributional solution to the equation $L\Gamma_y(\cdot) = \delta_y(\cdot)$ where δ_y is the Dirac mass concentrated at y. If L has constant coefficients then $\Gamma_y(x) = \Gamma_0(y - x)$ and the equation $Lu = f$ with f compactly supported distribution is solved (at least) by $u = \Gamma_0 * f$. One of the early successes of distribution theory was the following:

Theorem 2 *(Malgrange-Ehrenpreis, 1956) Every linear differential operator with constant complex coefficients has a (distributional) fundamental solution.*

Therefore, *all* linear differential operators *with constant coefficients* are locally solvable.

For operators with variable coefficients the situation is not so good. In 1957, just one year after Malgrange-Ehrenpreis' positive result, H. Lewy [13] found the first example of a linear equation with polynomial coefficients which may not have any solution: namely, the equation

$$\partial_{x_1} u + i\partial_{x_2} u - 2i(x_1 + ix_2)\partial_{x_3} u = f(x_1, x_2, x_3)$$

does not have any solution in any nonempty open set of $\mathbb{R}^3$, for some $f \in C^\infty(\mathbb{R}^3)$. Lewy's example was really striking for the time.[2] Through the years more examples of this kind were found. An interesting one is given by Mizohata's equation (quoted in [5, p. 84]):

$$\partial_x u + ix^k \partial_y u = f$$

which is solvable for k even, but not solvable for k odd (Grushin, 1971). An example of nonsolvable operator with real coefficients is the following in 3 variables (x, y, z), due to Hörmander (see [5, p. 85]):

$$Lu = \left(y^2 - z^2\right) u_{xx} + \left(1 + x^2\right)\left(u_{yy} - u_{zz}\right) - xy\, u_{xy} - (xy\, u)_{xy} + xz\, u_{xz} + (xz\, u)_{xz}.$$

A much easier and surprising example is the following, by Kannai, 1971 [9]: the operator

$$L = x\partial_{yy}^2 + \partial_x$$

is unsolvable at any point $(0, y_0)$; by contrast, the operator

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[2] Lewy's paper appeared on "Annals of Math.". In its review on the Math. Rev. we read: "Experience with linear partial differential equations has shown that they generally possess smooth local solutions provided the equations are sufficiently smooth. This paper produces the first example of a system with coefficients in C^∞ having no smooth solutions in any domain".

$$L = x\partial_{yy}^2 - \partial_x$$

is locally solvable at any point! The last example also shows that lower order terms are important for local solvability.

Let us now recall the definition of elliptic operator:

Definition 3 *A linear differential operator of order m (with complex coefficients)*

$$L = \sum_{|\alpha| \le m} a_\alpha(x)\, \partial_x^\alpha$$

is said elliptic if, given its principal symbol

$$p_0(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha$$

one has

$$p_0(x, \xi) = 0 \text{ if and only if } \xi = 0.$$

Then, we can state the following:

Theorem 4 *(See [12, Chap. 4]) Any elliptic operator with C^∞ coefficients is locally solvable at any point.*

In [5, pp. 85–90], several necessary or sufficient conditions for solvability are discussed. A criterion by Nirenberg-Treves gives a necessary and sufficient condition for the local solvability of the so-called *operators of principal type* ([5, pp. 85–90]). The definitive result in this direction is due to Beals-Fefferman, 1973 [1], but here we will not go into further details.

Let us now turn to the concept of *hypoelliptic operator* which, as we will see, is also related to that of solvable operator.

1.3 Hypoellipticity

Definition 5 *A differential operator L with $C^\infty(\Omega)$ coefficients (Ω open subset of $\mathbb{R}^n$) is said hypoelliptic in Ω if, for any open set $\Omega' \subset \Omega$ and any distribution $u \in D'(\Omega')$, $Lu \in C^\infty(\Omega') \Rightarrow u \in C^\infty(\Omega')$.*

Analogously, an operator is said analytic-hypoelliptic if $Lu \in C^\omega(\Omega) \Rightarrow u \in C^\omega(\Omega)$ ($C^\omega(\Omega)$ means analytic in Ω).[3]

[3] Analogously, one can introduce the notion of Gevray-hypoellipticity, replacing C^ω with Gevray classes. See [5, pp. 91–92].

1.3.1 Hypoelliptic Operators with Constant Coefficients

It follows immediately from the definition that if a hypoelliptic operator with constant coefficients possesses a fundamental solution Γ_0, then $\Gamma_0 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ (analogously, if the operator is analytic-hypoelliptic, any fundamental solution will be analytic outside the origin). A classical theorem by Schwartz states that the converse is true, too:

Theorem 6 *(See [18, Chap. 2, Theorem 2.1]) If the operator L with constant coefficients has a fundamental solution $C^\infty(\mathbb{R}^n \setminus \{0\})$ (or $C^\omega(\mathbb{R}^n \setminus \{0\})$), then L is hypoelliptic (respectively: analytic-hypoelliptic).*

For instance, in two variables we have:

Hypoelliptic and analytic hypoelliptic	Hypoelliptic but not analytic hypoelliptic	Not hypoelliptic
Laplacian: $\partial^2_{xx} + \partial^2_{yy}$ Cauchy-Riemann: $\partial_x + i\partial_y$	Heat[4]: $\partial_t - \partial^2_{xx}$	Wave: $\partial^2_{xx} - \partial^2_{yy}$ Schrödinger: $i\partial_t + \partial^2_{xx}$

Within the class of linear differential operators of any order m of *constant (complex) coefficients*, hypoelliptic operators have been characterized by Hörmander. The result is the following:

Theorem 7 *(See [4, p. 80]) Let*

$$L = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$$

(with a_α complex constants, m positive integer) and let $p(\xi)$ be the polynomial defined, via Fourier transform, by the relation

$$\widehat{Lu}(\xi) = p(\xi)\,\widehat{u}(\xi).$$

Then L is hypoelliptic if and only if

$$\lim_{|\xi| \to \infty} \frac{|\nabla p(\xi)|}{|p(\xi)|} = 0. \tag{1.1}$$

Example 8 *This criterion allows to check that Laplace equation, the heat equation, Cauchy-Riemann equation are hypoelliptic while the wave equation and Schrödinger equation are not. Less obvious examples are the following, which the reader can check by condition (1.1):*

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[4] Think to the fundamental solution of the heat equation, which vanishes identically for $t < 0$ but not for $t > 0$, hence is not analytic.