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Planar, Symmetric and Continuous Piecewise Linear Systems

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Jaume Llibre
Departament de Matemàtiques
Universitat Autònoma de Barcelona
Bellaterra, Barcelona
Spain

Antonio E. Teruel
Departament de Matemàtiques i Informàtica
Universitat de les Illes Balears
Palma-Illes Balears
Spain

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To Alba, Montse and Sara.

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Preface

Ordinary differential equations (ODEs) are the preferred language for the investigation and understanding of various natural phenomena. Employed extensively in natural sciences, engineering, and technology, ODEs are nowadays integrated in any standard undergraduate science curriculum, while continuing to be the subject of intensive research.

Although ODEs model a large number of natural phenomena, it is well known that not many admit explicit solution. For this reason, the qualitative theory and associated methods are often employed as an alternative investigative tool. When successful, the qualitative approach leads to a broader picture of important open subsets of solutions (sometimes the entire set), providing information about the ODEs' flow, parametric stability and bifurcations.

However, few families of ODEs allow a full treatment from the qualitative theory standpoint. The family of systems of linear differential equations is one of them. In the context of the qualitative theory, the importance of this family is evident when much of the local analysis of nonlinear ODEs is reduced to the study of their linear part. Nevertheless, this family exhibits limited richness from a dynamical systems standpoint.

In this book we consider planar systems of piecewise linear differential equations (PWLS), to which we apply the full program of the qualitative theory. PWLS may be considered as some of the most tractable nonlinear ODEs and they display a rich and interesting dynamical behaviour, comparable to that of general nonlinear ODEs.

Beyond the academic-theoretical significance, the study of PWLS has practical relevance. The interest in these class of systems is driven by concrete applications in engineering, in particular in control theory and the design of electric circuits.

This book is addressed to mathematicians, engineers, and scientists in general, who are interested in the qualitative theory of ODEs, PWLS in particular. It is also a reference book for anyone interested in the global phase portraits and the bifurcation sets of all the symmetric three-piece linear differential systems (here called *fundamental systems*), since their full characterization is presented here for the first time.

The book is divided into five chapters. Chapter 1 introduces fundamental systems, describes their global phase portraits (including behaviour at infinity) and the bifurcations occurring when parameters vary. To emphasize the importance of fundamental systems in applications, we discuss two well-known examples: the motor position control and the Wien bridge circuit. For the later and for specific values of the parameters, we describe the evolution of the phase portrait.

In Chapter 2 we collect the basic results of the qualitative theory of planar ODEs which are used in the rest of the book. To simplify the exposition of some concepts we have confined ourselves to ODEs having a complete flow. For this reason some of the results presented here are more restrictive than those that normally appear in the literature. In Section 2.5 we treat planar linear differential systems. We refer frequently to this section throughout the book. In Section 2.9 we formalize some aspects of the compactification of flows in order to apply this technique to the fundamental systems. As known, the Poincaré compactification is widely used in polynomial differential systems to study the behaviour of the flow near the infinity. However, although some differential equations can be compactified satisfactorily, we have not found a systematization of its use outside the class of polynomial differential systems.

Chapter 3 begins with the study of the fundamental systems. We show that within this class the existence and uniqueness theorem and the theorem on continuous dependence on initial conditions and parameters are valid. We further prove that the behaviour of these systems is determined by a pair of matrices, called fundamental matrices. This justifies that, except in very singular cases, we use the trace and the determinant of the two matrices as fundamental parameters to describe the dynamics of these systems. Additionally, we study the local phase portrait at the singular points, both finite and infinite, and we give some results about the existence and configuration of periodic orbits.

Poincaré maps of PWLS are determined by the linear differential systems which act in each of the pieces. For fundamental systems, one of these linear differential systems is homogeneous, while the other two are non-homogeneous. Consequently, in Chapter 4 we study all the Poincaré maps of linear differential systems associated to two cross sections. These cross sections are parameterized in such a way that the Poincaré maps become invariant under linear transformations. We note that the parametrization introduced here has important implications. First, it allows the study of the Poincaré maps by choosing, in each case, the simplest expression for the fundamental matrices. Usually we will assume that the matrices are expressed in their real Jordan normal form. Second, we can characterize the region in the parameter space where we can guarantee the existence of the Poincaré maps. Thus the bifurcation set associated to the non-existence of the Poincaré maps in the parameter space is an algebraic manifold homeomorphic to the Whitney umbrella. Finally, this parametrization establishes a link between Poincaré maps of PWLS and the class of differential systems which are called observable in control theory.

By collecting the results obtained in the previous chapters, in Chapter 5 we are able to describe and classify all the phase portraits of fundamental systems. The description of the phase portraits is carried out via the characterization of all separatrices and canonical regions. This allows us to use in a rigorous way the Marcus–Newmann–Peixoto Theorem on the topological classification of planar flows and to describe explicitly the bifurcation manifolds. Each of the sections of the chapter is devoted to fundamental systems having fixed the sign of two fundamental parameters. All sections of this chapter are structured similarly. First, we collect the results about singular points (both finite and infinite) and limit cycles. Second, we locate the rest of the separatrices of the system and we describe the behaviour of the canonical regions. Finally, we organize all the information in propositions which describe and classify fundamental systems when we vary the two parameters. At the end of each section we describe the bifurcations set and provide a picture of the parameter space representing the bifurcation manifolds and the corresponding phase portraits.

Readers interested only in such results can read the introductory Chapter 1 and then skip directly to Chapter 5, where they may find at the end of each section a complete list of phase portraits and their bifurcations.

The book has been organized in such a way so that the full classification of the global dynamics of the fundamental systems is obtained by using the qualitative theory of ODEs. Since there are many cases that must be considered, some propositions are very similar to each other and following all of them at the first reading becomes a little tedious. It may be recommended that at first reading only some of the proofs presented in Sections 3.11, 4.4 and 4.5 be followed in detail, so that the main arguments are understood. For instance, in Chapter 5, it may be useful to focus on one class of fundamental systems given by fixing the sign of the two fundamental parameters, and then follow the rest of the results in more detailed subsequent readings.

We thank Christina Stoica for her careful reading of the text of this book and her improvements to our poor English.

Jaume Llibre
Antonio E. Teruel
Barcelona, 2013.